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Forced oscillations of an enclosed rotating fluid

By P. G. BAINES
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

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The initial value problem related to axisymmetric forced oscillations of a rigidly rotating inviscid fluid enclosed in a finite circular cylinder is examined in linear approximation with the aid of the Laplace transform technique. An impulsive starting motion is considered. The solution consists of a 'periodic' motion which oscillates with the forcing frequency, together with a doubly infinite set of inertial modes whose presence is determined by the initial conditions and whose frequencies form a dense set in the range $(-2\omega, 2\omega)$, where ω is the angular velocity. The nature of the periodic or 'steady-state' part of the solution is strongly dependent on the precise value of the forcing frequency α $(\alpha > 0)$ when $\alpha \leqslant 2\omega$. In particular the system will resonate if α equals any one value of the dense set of resonant frequencies. It is shown that no internal sets of discontinuities in velocity or velocity gradient are present in the inviscid flow for finite times. Effects of viscosity on the inviscid solution are also discussed, and it is argued that when the inertial modes decay the steady-state flow will contain pseudo-random patterns of internal shear layers for some values of $\alpha < 2\omega$. It seems possible that these shear layers may be interpreted as owing their existence indirectly to viscosity.

1. Introduction

Properties of the motion of homogeneous rotating fluids have been studied by a number of authors, but nearly all the attention to date has been focused on bodies moving through or oscillating in an infinite fluid. Some of the results of these investigations have been surveyed by Morgan (1951) and Squire (1956). One of the most striking properties of forced oscillations in a rotating fluid is the difference between the types of motion obtained when the forcing frequency α is less than or greater than twice the angular velocity ω. When $\alpha > 2\omega$, under assumptions of small time-harmonic motion, the field equation for the disturbance pressure is elliptic in spatial variables, and the system has properties which are not markedly different from those of a non-rotating fluid. However, when $\alpha < 2\omega$ the field equation is hyperbolic in spatial variables, and disturbances may propagate through the fluid away from the source without losing intensity by means of inertial waves, which have frequencies in the range $(0, 2\omega)$. In some cases discontinuities may exist in the flow on characteristic surfaces whose presence is governed by the hyperbolic equation. Oser (1957, 1958) has made a theoretical and experimental study of such a case, namely an oscillating disk in an infinite rotating fluid. For further discussion see also Reynolds (1962a, b).

This paper aims to illuminate some of the corresponding properties of forced motion in a rotating fluid of finite extent. A special but typical case has been analysed by Wood (1965) under assumptions of steady motion. When $\alpha < 2\omega$ this analysis involves the solution of a boundary value problem where the field equation is hyperbolic. This problem is ill posed in that the flow solution obtained does not vary continuously with the boundary conditions, and the relation between solution and boundary conditions is in fact highly irregular. Any mathematical objections inherent in the formulation of such a problem are however overcome by considering an initial value problem.

The motion analysed here is that of an inviscid incompressible fluid in a finite right circular cylinder whose initially plane ends are impulsively set into simultaneous identical and axisymmetric time-harmonic deformations. The solution is obtained for all values of α and ω, including the case $\alpha = 2\omega$, for which no solution is obtained by steady-state assumptions. In every case the solution for the disturbance pressure (and hence the velocity) may be divided into two parts. The first part oscillates harmonically with the forcing frequency α, and is termed the 'steady-state' solution. For cases other than resonance this is the solution obtained by assuming the motion to be periodic in time, and it contains all the unusual properties mentioned above. The second part contains a double sum of normal modes each of which is spatially periodic in the direction of the axis of rotation, and whose frequencies form a dense set in the range $(-2\omega, 2\omega)$. It is shown that these modes combine with the steady-state solution in such a manner as to negate the anomalous properties of the latter, at least for finite times. Thus when a starting motion is considered irregularities present under steady-state assumptions are removed. The true properties of this flow, however, are difficult to determine in detail because of the complexity of the double sum.

When viscosity is included, as in every practical case, the normal modes will decay exponentially from the starting time (this has been proved for a general container by Greenspan (1965)). This will leave us with a forced motion which for large Taylor numbers may be expected to be a modification of the inviscid forced solution. Some aspects of the effects of viscosity on a similar motion have been studied by Wood (1966). His results show that at least in some cases, notably those where the corresponding inviscid motion has internal discontinuities in the velocity gradients, the inviscid steady-state solution is indeed a good approximation to the viscous one, with shear layers replacing surfaces of discontinuity in velocity gradient. It would appear then, paradoxically, that the inviscid steady-state solution only has significance for the viscous motion, at large times. In this sense the ensuing internal shear layers may be regarded as being indirectly created by viscosity.

Many of the anomalous properties of the steady-state solution as first noticed by Wood (1965) (considering the inviscid case for simplicity) are directly attributable to the infinity of the number of terms in the series which results from representation of the disturbance in terms of Bessel functions of the radial co-ordinate. The 'peculiar' features of the flow, namely the presence of finite discontinuities and logarithmic singularities in velocity gradient and velocity for various configurations, and their pseudo-random location, are only present if the forcing function $f(r)$ has a discontinuity in its derivative or has a non-zero gradient at the corner—i.e. $f'(a) \neq 0$ (although some discontinuities in velocity gradient may occur if only $f''(a) \neq 0$). For the purposes of discussion it is assumed that $f(r)$ is analytic and $f'(a) \neq 0$. For non-resonant cases where $\alpha < 2\omega$ the steady-state solution may be written in the form

$$S = \text{analytic terms} + \frac{\epsilon\kappa}{r^{\frac{1}{2}}}\sin\alpha t \sum_{m=M_0}^{\infty} \frac{(-)^m \cos(k_m r - \frac{1}{4}\pi)\sin(\Omega k_m az/l)}{m^3\cos(\Omega k_m a)}(1+O(1/m)), \tag{1.1}$$

where r and z are co-ordinates, k_m are the solutions of $J_1(k_m a) = 0$, κ is constant for a given configuration, a is the cylinder radius, l the cylinder length, M_0 is a sufficiently large integer, and

$$\Omega = \frac{\alpha l}{(4\omega^2-\alpha^2)^{\frac{1}{2}}a}. \tag{1.2}$$

It appears that a solution of this form is obtained whenever the cylinder is subjected to a periodic three-dimensional disturbance. For some values of Ω each term in the series is of $O(1/m^3)$ and for others $O(1/m^2)$. Since $k_m \sim m\pi$ for m large, discontinuities may then occur respectively in the second and first derivatives of S and therefore in the velocity gradients and velocities. The existence of these and other phenomena is critically dependent on the basic parameters. For example, an infinitesimal change in any of α, ω, l or a could change the motion from a resonant state to a pseudo-random pattern of velocity discontinuities, or vice versa. However, if the disturbance is representable as a finite series of Bessel functions, no discontinuities will be present.

The phenomenon may be interpreted in terms of a superposition of stationary inertial waves, all having the same frequency but different wavelengths, and with their wave-fronts parallel to the characteristic cones. These waves will reflect off the side wall at the characteristic angle. The motion in the interior is representative of the forcing function $f(r)$ extended along characteristics, and on opposite sides of the characteristics emanating from the corners we expect the disturbance to be symmetric (in the neighbourhood) by virtue of the side-wall reflexion of the constituent inertial waves. Hence if $f'(a) \neq 0$ there is a virtual discontinuity in $f'(r)$ at $r = a$, and this will be reproduced in the interior as a discontinuity in the velocity gradient along the corner characteristics.

With the exception of §5, all statements following refer to inviscid motion unless explicitly stated otherwise. Nonlinear effects have been neglected, although in the case of resonance this cannot be valid for large times.

2. Formulation

We consider the motion of an inviscid incompressible fluid enclosed in a finite circular cylinder which is rotating with constant angular velocity ω about its axis of symmetry. The ends are initially plane and perpendicular to the axis and the fluid is in a state of rigid rotation. At time $t = 0$ the ends are set impulsively into small time-harmonic oscillations in the axial direction. The deformations are axisymmetric and in phase.

Let the cylinder have length $2l$ and radius a. We take cylindrical polar coordinates (r, ϕ, z), with the axis of the cylinder along the z-axis and the ends initially in the planes $z = \pm l$. Let the ends have the motion

$$\left.\begin{aligned} z = \pm l \quad & (t < 0), \\ \pm l + \epsilon f(r) \sin \alpha t \quad & (t \geqslant 0), \end{aligned}\right\} \tag{2.1}$$

where $\epsilon \ll l$, α is a constant and $f(r)$ is a sufficiently regular function of r. Consider axes rotating with the cylinder. If we let the velocity components in the direction of increasing (r, ϕ, z) be (u, v, w), the linearized boundary condition on the fluid at both $z = \pm l$ is, for $t \geqslant 0$,

$$w = \epsilon \alpha f(r) \cos \alpha t. \tag{2.2}$$

At $r = a$ the boundary condition is

$$u = 0. \tag{2.3}$$

The conditions for impulsive motion (Lamb 1932, §11) give for $t = 0$ (or more correctly $t \to 0+$)

$$\mathbf{u} = -\nabla \Pi(r, z), \tag{2.4}$$

where $\mathbf{u}$ is the velocity vector (u, v, w) and $\Pi(r, z)$ is an impulsive pressure function which must satisfy

$$\nabla^2 \Pi = 0. \tag{2.5}$$

Expressing the boundary conditions at $t = 0$ in terms of Π gives

$$\left.\begin{aligned} r = a: \quad & \frac{\partial \Pi}{\partial r} = 0, \\ z = \pm l: \quad & \frac{\partial \Pi}{\partial z} = -\epsilon \alpha f(r). \end{aligned}\right\} \tag{2.6}$$

We thus have a simple boundary value problem for Π which has the solution

$$\Pi(r, z) = -\epsilon \frac{2\alpha}{a^2} \sum_{m=1}^{\infty} \frac{\int_0^a r f(r) J_0(k_m r)\, dr}{k_m J_0^2(k_m a)} J_0(k_m r) \frac{\sinh k_m z}{\cosh k_m l}, \tag{2.7}$$

where the k_m are the solutions of

$$J_1(k_m a) = 0. \tag{2.8}$$

Equation (2.4) then gives the velocity distribution at $t = 0$ which constitutes the initial conditions for the subsequent motion.

The linearized Euler equations relative to the rotating axes are

$$\frac{\partial u}{\partial t} - 2\omega v = -\frac{\partial \chi}{\partial r}, \quad \frac{\partial v}{\partial t} + 2\omega u = 0, \quad \frac{\partial w}{\partial t} = -\frac{\partial \chi}{\partial z}, \tag{2.9}$$

where $\chi(r, z, t)$ is the disturbance pressure, absorbing the centrifugal and gravitational factors, and ϵ (equation (2.1)) is supposed sufficiently small so that nonlinear terms may be neglected. From dimensional considerations this will be so if

$$\epsilon \ll \frac{\omega l}{\alpha |f(r)|_{\max}}, \tag{2.10}$$

where $|f(r)|_{\max}$ is the maximum value of $|f(r)|$.† The continuity equation is

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0, \tag{2.11}$$

and equations (2.9) and (2.10) give

$$\frac{\partial^2}{\partial t^2}(\nabla^2\chi)+4\omega^2\frac{\partial^2\chi}{\partial z^2} = 0. \tag{2.12}$$

This is the basic field equation for χ, and is to be solved such that the boundary and initial conditions are satisfied. The boundary conditions (2.2, 2.3) in terms of χ are

$$r = a: \frac{\partial \chi}{\partial r} = 0, \tag{2.13}$$

$$z = \pm l: \quad \frac{\partial \chi}{\partial z} = \epsilon\alpha^2 f(r)\sin\alpha t. \tag{2.14}$$

By use of the relations

$$\nabla^2\chi = 2\omega\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right), \quad \frac{\partial}{\partial t}(\nabla^2\chi) = 4\omega^2\frac{\partial w}{\partial z}, \tag{2.15}$$

we have the initial conditions at $t = 0+$

$$\nabla^2\chi(r,z,t) = 0, \tag{2.16}$$

$$\frac{\partial}{\partial t}(\nabla^2\chi) = -4\omega^2\frac{\partial^2\Pi}{\partial z^2}, \tag{2.17}$$

which are sufficient to determine χ. In conjunction with the boundary conditions, (2.15) implies that we may take $\chi = 0$ at $t = 0$.

3. Formal solution

To solve (2.11) by the Laplace transform procedure, we define

$$X(r,z,s) = \mathscr{L}\chi(r,z,t) = \int_0^\infty e^{-st}\chi(r,z,t)\,dt, \tag{3.1}$$

where $\operatorname{Re} s > 0$. The transformed field equation for $X(r,z,s)$ is then

$$\frac{\partial^2 X}{\partial r^2}+\frac{1}{r}\frac{\partial X}{\partial r}+\lambda^2\frac{\partial^2 X}{\partial z^2} = -\frac{4\omega^2}{s^2}\frac{\partial^2\Pi}{\partial z^2}, \tag{3.2}$$

where $\lambda^2 = 1+(2\omega/s)^2$. The transformed boundary conditions are

$$r = a: \quad \frac{\partial X}{\partial r} = 0, \tag{3.3}$$

$$z = \pm l: \quad \frac{\partial X}{\partial z} = \epsilon\frac{\alpha^3}{\alpha^2+s^2}f(r). \tag{3.4}$$

† However this cannot be uniformly valid for large times, in view of resonances in the solution and the possibility of nonlinear resonant interactions.

A particular solution of (3.2) is readily found to be

$$X_1 = -\Pi(r,z). \tag{3.5}$$

The general solution of the homogeneous equation is

$$X_2 = \sum_{m=1}^{\infty} B_m J_0(k_m r)\sinh(k_m z/\lambda), \tag{3.6}$$

where B_m are independent of r, z and the k_m are again the solutions of

$$J_1(k_m a) = 0.$$

We need to choose the B_m such that

$$X = X_1 + X_2$$

satisfies the boundary condition (3.4). This will be the case if

$$B_m = -\epsilon 2\alpha \frac{\int_0^a rf(r)\, J_0(k_m r)\, dr}{a^2 k_m J_0^2(k_m a)} \frac{\lambda}{\cosh k_m l/\lambda}\left(1-\frac{\alpha^2}{\alpha^2+s^2}\right), \tag{3.7}$$

so that

$$X(r,z,s) = \epsilon\frac{2\alpha}{a^2}\sum_{m=1}^{\infty}\frac{\int_0^a rf(r)\, J_0(k_m r)\, dr}{k_m J_0^2(k_m a)} J_0(k_m r) \times\left[\frac{\sinh k_m z}{\cosh k_m l}-\lambda\left(1-\frac{\alpha^2}{\alpha^2+s^2}\right)\frac{\sinh k_m z/\lambda}{\cosh k_m l/\lambda}\right]. \tag{3.8}$$

The solution for χ is now given by the inverse transform

$$\chi(r,z,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\chi(r,z,s)\, ds, \tag{3.9}$$

where $c > 0$. In order to obtain this solution we need to consider four different cases (without losing generality we may take α and ω to be non-negative).

(i) $\alpha > 2\omega$

The integrand has simple poles at the points $s = \pm i\alpha$ and also at the points $s = \pm i\beta_{mn}$, where

$$\beta_{mn} = \frac{2\omega(2n+1)\pi}{[4k_m^2 l^2 + (2n+1)^2\pi^2]^{\frac{1}{2}}}, \tag{3.10}$$

for all integers n and all positive integers m. Clearly, $|\beta_{mn}| < 2\omega$ for all m, n, and the set of poles for each m has limit points at $\pm 2i\omega$. The function has an essential singularity at each of these two points, and to avoid a discussion of these singularities we adopt the procedure of evaluating the residues at $s = \pm i\alpha$, $\pm i\beta_{mn}$, and then verify that their sum is the desired solution for χ. The sum of the residues of

$$\lambda\left(1-\frac{\alpha^2}{\alpha^2+s^2}\right)\frac{\sinh k_m z/\lambda}{\cosh k_m l/\lambda}e^{st} \tag{3.11}$$

at $s = i\alpha$ and $s = -i\alpha$ is

$$-\frac{(\alpha^2-4\omega^2)^{\frac{1}{2}}}{\alpha^2}\frac{\sinh\gamma_m z}{\cosh\gamma_m l}\sin\alpha t, \tag{3.12}$$

where
$$\gamma_m = \frac{k_m\alpha}{(\alpha^2-4\omega^2)^{\frac{1}{2}}}. \tag{3.13}$$

The sum of the residues at $s = i\beta_{mn}, -i\beta_{mn}$ is

$$-\omega(4k_m l)^3 A_{mn}(-)^n \sin(2n+1)\frac{\pi z}{2l}\sin\beta_{mn}t, \tag{3.14}$$

where

$$A_{mn} = \frac{1}{(2n+1)\pi[4k_m^2 l^2+(2n+1)^2\pi^2]^{\frac{3}{2}}} - \frac{\alpha^2}{(2n+1)\pi[4k_m^2 l^2+(2n+1)^2\pi^2]^{\frac{1}{2}}[4k_m^2 l^2\alpha^2+(\alpha^2-4\omega^2)(2n+1)^2\pi^2]}. \tag{3.15}$$

Hence for $\alpha > 2\omega$ we have

$$\chi(r,z,t) = \epsilon\sum_{m=1}^{\infty} C_m J_0(k_m r)\left\{(\alpha^2-4\omega^2)^{\frac{1}{2}}\frac{\sinh\gamma_m z}{\cosh\gamma_m l}\sin\alpha t - \omega(4k_m l)^3\sum_{n=0}^{\infty}(-)^n A_{mn}\sin(2n+1)\frac{\pi z}{2l}\sin\beta_{mn}t\right\}, \tag{3.16}$$

where
$$C_m = \frac{2\alpha}{a^2 k_m J_0^2(k_m a)}\int_0^a rf(r)\,J_0(k_m r)\,dr. \tag{3.17}$$

(ii) $\alpha < 2\omega, \quad \alpha \neq \beta_{mn}$

The same procedure as in case (i) applies to give the same expression for χ. The solution has vastly different properties in this case however, and is better written in the form

$$\chi(r,z,t) = \epsilon\sum_{m=1}^{\infty} C_m J_0(k_m r)\left\{(4\omega^2-\alpha^2)^{\frac{1}{2}}\frac{\sin\alpha_m z}{\cos\alpha_m l}\sin\alpha t - \omega(4k_m l)^3\sum_{n=0}^{\infty}(-)^n A_{mn}\sin(2n+1)\frac{\pi z}{2l}\sin\beta_{mn}t\right\}, \tag{3.18}$$

where
$$\alpha_m = \frac{k_m\alpha}{(4\omega^2-\alpha^2)^{\frac{1}{2}}}.$$

It may readily be verified that both (3.16) and (3.18) satisfy all the conditions, by virtue of the relation

$$\frac{\sin\alpha_m z}{\cos\alpha_m l} = 8\alpha_m l\sum_{n=0}^{\infty}\frac{(-)^n\sin(2n+1)(\pi z/2l)}{(2n+1)^2\pi^2-4\alpha_m^2 l^2} \quad (-l \leqslant z \leqslant l), \tag{3.19}$$

and the corresponding expressions for

$$\frac{\sinh k_m z}{\cosh k_m l}, \quad \frac{\sinh\gamma_m z}{\cosh\gamma_m l}.$$

(iii) $\alpha = \beta_{mN}$

$X(r,z,s)$ now has double poles at $s = \pm i\alpha$. Summing all the residues in this case gives

$$\chi(r,z,t) = -\epsilon \sum_{m=1}^{\infty} C_m J_0(k_m r) \left\{ a_1 \sin\left[(2N+1)\frac{\pi z}{2l}\right] t\cos\alpha t + a_2 \frac{z}{l} \cos(2N+1)\frac{\pi z}{2l} \sin\alpha t + a_3 \sin(2N+1)\frac{\pi z}{2l}\sin\alpha t + \omega(4k_m l)^3 \sum_{\substack{n=0\\ n\neq N}}^{\infty} (-)^n A_{mn} \sin(2n+1)\frac{\pi z}{2l}\sin\beta_{mn} t \right\}, \tag{3.20}$$

where a_1, a_2, a_3 are the constants

$$a_1 = \frac{(-)^N(4\omega^2-\alpha^2)^2}{4\omega^2 k_m l}, \quad a_2 = (-)^N(4\omega^2-\alpha^2)^{\frac{1}{2}}, \quad a_3 = \frac{(-)^N(4\omega^2-\alpha^2)(2\omega^2-3\alpha^2)}{4\omega^2 k_m l\alpha}. \tag{3.21}$$

This solution also satisfies the field equation, boundary and initial conditions.

(iv) $\alpha = 2\omega$

Here the above poles at $s = \pm i\alpha$ disappear respectively into the essential singularities $s = \pm 2i\omega$, and the solution consists of the sum of the normal modes only, in the form

$$\chi(r,z,t) = -\epsilon\omega \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} (-)^n (4k_m l)^3 C_m A_{mn} J_0(k_m r)\sin(2n+1)\frac{\pi z}{2l}\sin\beta_{mn} t, \tag{3.22}$$

where in this case

$$A_{mn} = -\frac{1}{(2n+1)\pi[4k_m^2 l^2+(2n+1)^2\pi^2]^{\frac{1}{2}}}\left(\frac{1}{4k_m^2 l^2} - \frac{1}{4k_m^2 l^2+(2n+1)^2\pi^2}\right).$$

At first sight this solution may not appear to satisfy the boundary conditions at $z = \pm l$, since $\cos(2n+1)(\pi z/2l)$ will be zero there. However, let us consider one Bessel component, i.e. a case where $f(r) = J_0(k_m r)$. Then

$$\chi(r,z,t) = -\epsilon\, 8(4\omega)^2 k_m^2 l^3 J_0(k_m r) \sum_{n=0}^{\infty} (-)^n A_{mn} \sin(2n+1)\frac{\pi z}{2l}\sin\beta_{mn} t,$$

and if we consider the series $\partial\chi/\partial z$ for any given t we find that as $|z| \to l-$

$$\frac{\partial\chi}{\partial z} \to \epsilon\, 4\omega^2 f(r)\sin 2\omega t, \tag{3.23}$$

as required. The discontinuity at the end-points may be regarded as a manifestation of the Fourier series representation of the function. It should be noted that in taking the limit of $\partial\chi/\partial z$ as $\alpha \to 2\omega+$, where χ is the complete solution (equation (3.16)), the boundary conditions at $z = \pm l$ remain satisfied by virtue of non-uniformity over z of the limit of each of the two parts of the solution (steady state and modes) in the neighbourhoods of the ends of the cylinder. This suggests that in the limit $|z| \to l-$ the full series in m (equation (3.22)) satisfies equation (3.23), although the limit is non-uniform in m (and also in α).

4. Discussion of the solution

The solutions for the disturbance pressure function χ enable us to obtain the velocities u, v, w. By use of the identities (2.14) we have

$$\frac{\partial}{\partial r}(rv) = \frac{r}{2\omega}\nabla^2\chi, \quad \frac{\partial w}{\partial z} = \frac{1}{4\omega^2}\nabla^2\chi_t, \tag{4.1}$$

where $\chi_t = \partial\chi/\partial t$, and these together with (2.9) give

$$\left.\begin{aligned} u &= -\frac{1}{4\omega^2 r}\int^r r\nabla^2\chi_t dr, \\ v &= \frac{1}{2\omega r}\int^r r\nabla^2\chi\, dr, \\ w &= \frac{1}{4\omega^2}\int^z \nabla^2\chi_t dz. \end{aligned}\right\} \tag{4.2}$$

Hence the velocities are effectively linear functions of the first derivatives of the pressure.

We now consider the consequences of the various solutions for χ. When $\alpha > 2\omega$, χ has the form (equation (3.16))

$$\chi(r,z,t) = -\epsilon(\alpha^2-4\omega^2)^{\frac{1}{2}}\sum_{m=1}^{\infty} C_m J_0(k_m r)\frac{\sinh\gamma_m z}{\cosh\gamma_m l}\sin\alpha t + \text{normal inertial modes}.$$

With the exception of the inertial modes, this is akin to the irrotational surface wave solution. For the radial and azimuthal velocity components for the forced modes, most of the motion is near the ends of the cylinder with exponential decay towards the centre. This case presents little of interest in comparison with that of $\alpha < 2\omega$.

When $\alpha < 2\omega$, $\alpha \neq \beta_{mn}$, χ has the form (equation (3.18))

$$\chi(r,z,t) = -\epsilon(4\omega^2-\alpha^2)^{\frac{1}{2}}\sum_{m=1}^{\infty} C_m J_0(k_m r)\frac{\sin\alpha_m z}{\cos\alpha_m l}\sin\alpha t + \text{modes}.$$

This solution except for the modes will hereafter be denoted S, and is that obtained under the assumption of steady oscillating motion with $\sin\alpha t$ dependence in χ. The inertial modes are present in order to satisfy the initial conditions, and may be regarded as generated by the starting motion. S has unusual properties which may be seen by considering the terms of the series when m is large. Assuming that the deformations of the ends are smooth we may take $r^{\frac{1}{2}}f''(r)$ to be continuous in $[0,a]$ and $f'(0) = 0$, where

$$f'(r) = \frac{d}{dr}f(r).$$

Without losing generality we may also take $f(a) = 0$, and we then have for k_m large (Watson 1944)

$$\begin{aligned}\int_0^a rf(r)\, J_0(k_m r)\, dr &= af'(a)\frac{J_0(k_m a)}{k_m^2} + O\left(\frac{1}{k_m^{\frac{7}{2}}}\right) \\ &= \left(\frac{2a}{\pi}\right)^{\frac{1}{2}}\frac{f'(a)}{k_m^{\frac{5}{2}}}\cos(k_m a - \tfrac{1}{4}\pi) + O(1/k_m^{\frac{7}{2}}).\end{aligned} \tag{4.3}$$

Also, for m large we have

$$k_m a = (m - m_0 + \tfrac{1}{4})\,\pi - \frac{3}{8\pi m} + O\left(\frac{1}{m^2}\right), \tag{4.4}$$

where m_0 is some integer. Using the asymptotic form for $J_0(k_m r)$ and excluding r from a neighbourhood of zero we may write

$$S = \text{analytic terms} + \epsilon\,\frac{\kappa \sin \alpha t}{r^{\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{(-)^m \cos(k_m r - \frac{1}{4}\pi)\sin(\Omega k_m a z/l)}{m^3 \cos(\Omega k_m a)}\left(1 + O\left(\frac{1}{m}\right)\right), \tag{4.5}$$

where κ is a constant and

$$\Omega = \frac{\alpha l}{(4\omega^2 - \alpha^2)^{\frac{1}{2}}\,a}. \tag{4.6}$$

When Ω is rational, i.e. $\Omega = P/Q$, where P and Q are coprime integers, an inspection of the series reveals that S has discontinuities in the second derivative and in some cases, depending on the relative values of P and Q, discontinuities in the first derivative (this analysis will not be pursued here, and the reader is referred to Wood (1965) and Baines (1966) for the details for similar cases). These discontinuities will, in the absence of the modes, be realized as discontinuities in the velocity components and their gradients across the cones

$$\frac{r}{a} \pm \Omega \frac{z}{l} = \text{constant}, \tag{4.7}$$

which emanate from the top and bottom corners of the cylinder, and across the continuation of these cones by reflexion at the side, ends and axis of the cylinder. These cones may be identified as characteristics, and will henceforth be referred to as such. When Ω is rational these reflected characteristic cones will return to the top or bottom corners after a finite number of reflexions. However, if Ω is irrational there is no terminating point for the reflexions, and these cones are dense throughout the cylinder. Also, the argument which leads to the discontinuities in u, v, w and their gradients fails when Ω is irrational, and the corresponding properties of the motion in this case are not known except for that of resonance. Hence if the disturbance pressure is purely represented by S the corresponding flow in the cylinder has peculiar properties which are strongly dependent on Ω, such that an infinitesimal change in Ω could drastically change the characteristic pattern and hence also the flow field. The situation is further complicated by the fact that the resonance frequencies β_{mn} are dense throughout the range $(0, 2\omega)$. This will be commented on later.

We now consider the full solution for χ, equation (3.18). Using the power series expansions for $\sin \alpha t$, $\sin \beta_{mn} t$ we may express χ in the convergent expansion

$$\chi(r,z,t) = \chi_1(r,z)\,t + \chi_2(r,z)\,t^3 + \ldots + \chi_s t^{2s-1} + \ldots, \tag{4.8}$$

where

$$\chi_s = \epsilon \sum_{m=1}^{\infty} C_m J_0(k_m r)\left\{(4\omega^2 - \alpha^2)^{\frac{1}{2}} \frac{\sin \alpha_m z}{\cos \alpha_m l}\alpha^{2s-1} - \omega(4k_m l)^3 \sum_{n=0}^{\infty} (-)^n A_{mn} \beta_{mn}^{2s-1} \sin(2n+1)\frac{\pi z}{2l}\right\}\frac{1}{(2s-1)!}. \tag{4.9}$$

Since
$$\nabla^2 \chi_t|_{t=0} = -4\omega^2 \Pi_{zz},$$
we have
$$\nabla^2 \chi_1 = -4\omega^2 \Pi_{zz}. \tag{4.10}$$

This is an elliptic equation for χ_1 which can therefore have none of the discontinuity and pseudo-randomness properties of S. Substituting (4.8) in the field equation (2.11) and equating coefficients of powers of t, we obtain

$$\left.\begin{aligned} 6\nabla^2\chi_2 &= -4\omega^2 \chi_{1,zz},\\ 20\nabla^2\chi_3 &= -4\omega^2\chi_{2,zz},\\ &\ \vdots\\ (2s-1)(2s-2)\nabla^2\chi_s &= -4\omega^2\chi_{s-1,zz},\\ &\ \vdots\\ \cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{4.11}$$

Hence every function χ_s satisfies a Poisson equation, and it follows that at no time does the function χ possess first or second derivative discontinuities in the interior, foı any value of α. Also χ must vary continuously with the boundary conditions and dimensions of the cylinder, at least for finite times. The doubly infinite set of normal modes effectively combines with S to negate the anomalous properties of that part of the solution. For the cases $\alpha = \beta_{mn}$, $\alpha > 2\omega$ and $\alpha = 2\omega$ the same argument will apply. Thus for all values of α the functions χ_s and χ will be analytic in the interior of the cylinder for finite times.

When $\alpha = \beta_{mN}$ we have the case where the forcing frequency is equal to one of the resonant frequencies of the system. Equation (3.20) shows that there are essentially three terms of frequency α, one of which has the form

$$\sin\{(2N+1)(\pi z/2l)\}t\cos\alpha t,$$

and will oscillate with large amplitude when t is large. The resonant frequencies β_{mN} are dense in the range $[0, 2\omega]$ in the same manner in which rational numbers are dense in the field of real numbers, and the corresponding solutions for χ will be similarly placed among the non-resonant solutions. The above result (equation (4.11)), however, still requires that χ vary continuously with α, ω, a and l for finite times. We may therefore expect the (continuous) variation of amplitude of χ with α, etc., to become larger as t increases and the

$$\sin\{(2N+1)(\pi z/2l)\}\ t\cos\alpha t$$

terms become dominant. For the other normal modes the A_{mn} may be written

$$\underset{n\neq N}{A_{mn}} = \frac{1}{(2n+1)\pi[4k_m^2 l^2 + (2n+1)^2\pi^2]^{\frac{1}{2}}} - \frac{(2N+1)^2\pi^2}{4k_m^2 l^2(2N+1)\pi[4k_m^2 l^2+(2n+1)^2\pi^2]^{\frac{1}{2}}[(2N+1)^2\pi^2-(2n+1)^2\pi^2]}. \tag{4.12}$$

When $\alpha = 2\omega$ we have the singular case which exists for all situations involving forced oscillations in rotating fluids. For the conditions given here, no solution of (2.11) oscillating harmonically with frequency 2ω is possible, and the inertial modes constitute the whole solution as given by (3.22). The characteristic cones of the case $\alpha < 2\omega$ have degenerated into flat surfaces $z =$ constant, and those emanating from the corners have become coincident with the end surfaces $z = \pm l$.

It is possible that this motion ($\alpha = 2\omega$) is physically unstable, and some experiments carried out by the author do not contradict this suggestion.

5. Nonlinear and viscous effects

The number of possible patterns of behaviour of the system multiplies when nonlinear effects are considered. Linearization has been effected at two points—first in the neglect of the $\mathbf{u}.\nabla\mathbf{u}$ terms in the field equation, and secondly in the end conditions via equation (2.2). In an investigation of the behaviour of the system to second order in ϵ both of these features need to be considered, although their effects are additive. In both cases the second-order contribution to the motion is dependent on the first-order solution, and, if only the forced modes (of frequency α) are considered, both effects will produce a second-order contribution to the pressure function χ which is out of phase with the first-order term and has frequency 2α, provided $\alpha < \omega$. In addition the nonlinear term in the equation of motion will produce a time-independent second-order contribution to χ, resulting in a steady azimuthal drift which will vary with co-ordinates r and z.

The possibility of second-order resonant interactions among the multitude of first-order modes, both free and forced, cannot be ruled out. The most interesting possibility is perhaps the interaction of two forced modes to produce free modes. A cursory examination of the resonance conditions for this case indicates that they could be satisfied for various values of the frequency ratio α/ω and the cylinder length $2l$. A detailed discussion of resonant interactions among these 'cylindrical waves' does not however seem justified at present in view of the very unusual features, including resonances, of the first-order solution.

We now consider the possible effects of viscosity on the linear solution, neglecting the above-mentioned nonlinear factors. On this basis viscosity will affect the forced and free modes independently, and since the free inertial modes are only generated by the impulsive commencement of motion and are not regenerated by any linear process, they would be expected to decay viscously on simple energy considerations. This process has been discussed at length by Greenspan (1964, 1965), and his results show that a given set of free inertial modes will decay exponentially in the uniform time scale

$$t = O(R^{\frac{1}{2}}/\omega),$$

where the Taylor number $R = \omega l^2/\nu$. Hence after a period of time corresponding to this viscous time scale, the only disturbance remaining in the flow (on linear theory) will be the viscous modification of the forced solution, S.

Some of the effects of viscosity on a similar type of forced motion have been studied by Wood (1966) under assumptions of steady motion. His system consists of a similar circular cylinder to the one considered here, but with rigid plane ends. The disturbance is created by forcing the cylinder to precess about an axis inclined at a small angle to its axis of rotation and symmetry, and the source of disturbance is thus manifested in the equations of motion as a 'virtual' body force. This contrasts with the forced surface displacement (equation (2.1)) considered in this work, and because of this difference Wood's lengthy viscous analysis cannot be directly tailored to this particular system. However, with the exception of one or two minor details the inviscid solutions for the two cases are very similar, and have the same properties. In view of this it seems not unreasonable to suppose that the basic conclusions of the viscous analysis will carry over. (The dubious reader may prefer to accept the converse: that the commencement of motion in Wood's cylinder would generate a similar set of normal inertial modes.) Wood's viscous analysis is only applicable to non-resonant cases where discontinuities and logarithmic singularities appear in the (inviscid) velocity gradient, and not in the velocity—cases corresponding to Ω rational here, with $\Omega = P/Q$, where P and Q are coprime integers and with $P \not\equiv 2 \pmod 4$. With Ω restricted to this set of values then, and with the above assumption, Wood's results state that the inviscid surfaces of logarithmic singularity will be replaced by shear layers of thickness $O(R^{-\frac{1}{3}})$ in which the velocity gradients are $O(R^{\frac{1}{6}})$ larger than elsewhere. The extreme sensitivity of the internal shear layer pattern to the value of the parameter Ω remains. The reader is referred to his paper for further details. Viscosity is therefore responsible for the very existence of these shear layers, whose inviscid counterparts are not present when a starting motion is considered.

Another possibility is that, together with the above behaviour, resonant modes with frequencies near the forcing frequency will grow linearly with time and render the motion unstable.

The situation $\alpha = 2\omega$ represents a special case, as the steady-state solution is zero in the interior and the free inertial modes must decay in the same time as in other cases. The linearized boundary condition (2.2) is clearly no longer satisfactory as a representation of an oscillating surface; whereas, if the imposed normal velocity distribution is taken as the true boundary condition, the disposal of the injected fluid with no motion in the interior suggests some unusual boundary-layer features.

A careful and as yet unpublished experimental study of forced oscillations in a rotating circular cylinder is currently being carried out by A. D. McEwan at Aeronautical Research Laboratories, Fisherman's Bend, Australia. Many of the features of the theory outlined and quoted here have been reproduced. In particular the existence of shear layers emanating from cylinder corners, their sensitivity to container dimensions and the presence of resonances at theoretically predicted configurations have been confirmed.

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